Journal of Statistical Physics, Vol. 20, No. 4, 1979

Stochastic Dynamics of Two-Dimensional Infinite-Particle Systems

J. Fritz[1]

Received May 22, 1978; revised November 7, 1978

The time evolution of an open system of infinitely many two-dimensional classical particles is investigated. Particles are interacting by a singular pair potential U, and each particle is connected to a heat bath of temperature T. The heat baths are represented by independent white noise forces and Langevin damping terms. Existence of strong solutions to the corresponding infinite system of stochastic differential equations is proved for initial configurations with a logarithmic order of energy fluctuations. Gibbs states for U at temperature T are invariant under time evolution.

KEY WORDS: Infinite systems; nonequilibrium dynamics; stochastic differential equations; Gibbs states.

1. INTRODUCTION

The aim of this paper is to initiate the study of certain random perturbations of two-dimensional nonequilibrium dynamics. The methods of Refs. 3 and 4 are developed further in order to obtain the existence of strong solutions to the following infinite system of stochastic differential equations. Consider an infinite configuration $\omega = \{(x_i, v_i); i \in I\}$ of two-dimensional labeled particles interacting by a pair potential $U = U(x)$; $x_i = x_i(\omega)$ and $v_i = v_i(\omega)$ denote the position and the velocity of the ith particle; I is the set of positive integers. Particles are assumed to be of unit mass, and in addition to the conservative interparticle forces $-\text{grad } U(x_i - x_j)$, $j \neq i$, the nonconservative force $-\lambda v_i$ and a white noise force are acting on the ith particle. Then the stochastic differential equations of motion are

$$\begin{aligned} dv_i &= -\sum_{j \neq i} \text{grad } U(x_i - x_j)\, dt - \lambda v_i\, dt + \sigma\, dw_i \\ dx_i &= v_i\, dt, \qquad i \in I \end{aligned} \tag{I}$$

[1] Mathematical Institute, Budapest, Hungary.

0022-4715/79/0400-0351$03.00/0

where $w_i = w_i(t)$ is a sequence of independent, standard, two-dimensional Wiener processes, and λ and σ are nonnegative constants. We shall show that (I) generates a Markov time evolution in the space Ω_0 of infinite configurations with a logarithmic order of energy fluctuation. This stochastic dynamics can be interpreted as the time evolution of a large classical system connected to a heat bath of temperature $\sigma^2/2\lambda$. Indeed, a canonical Gibbs state for U at temperature T is time-invariant if and only if $T = \sigma^2/2\lambda$. If λ and σ go to zero, then the stochastic dynamics converges to the classical dynamics, which was constructed in Ref. 4.

2. PRELIMINARIES

First we have to clarify the meaning of (I). Let $\mathbf{R}^2$ denote the two-dimensional Euclidean space with the usual norm $|\cdot|$ and scalar product $(.,.)$; $\mathbf{Z}^2$ is the integer lattice in $\mathbf{R}^2$. The interaction potential U is assumed to be a continuously differentiable function $U = U(x)$, $x \in \mathbf{R}^2$, $x \neq 0$, such that $U(x) = U(-x)$, $\lim_{x \to 0} U(x) = +\infty$ and $U(x) = 0$ if $|x| \geqslant R$; $R < +\infty$ is the range of U. To prove existence of solutions we need the following regularity conditions for U; they are the same as in Ref. 4. There exist positive constants a, b, c, d, δ, L such that: (a) $U(x) > 0$ if $|x| \leqslant \delta$,

$$|x|\,|\text{grad } U(x)| \leqslant a + bU(x) \tag{E}$$

(b) at least one of

$$|\text{grad } U(x)|^2 \leqslant cU(x) \qquad \text{if} \quad |x| \geqslant \delta \tag{P}$$

and

$$cU(x) \geqslant |x|^{-4} \qquad \text{if} \quad |x| \leqslant \delta \tag{R}$$

holds, and (c) $|U(x)| \leqslant u$ and $|U(y)| \leqslant u$ imply that

$$|\text{grad } U(x) - \text{grad } U(y)| \leqslant |x - y| L(1 + u)^d \tag{U}$$

The validity of (E), (U), and one of (P) and (R) will be assumed throughout this paper. For a discussion of these conditions see Refs. 3 and 4.

The configuration space Ω is defined as the set of locally finite labeled configurations $\omega = \{(x_i, v_i); i \in I\}$, where $x_i = x_i(\omega)$ and $v_i = v_i(\omega)$ are two-dimensional vectors, and the sequence $x_i(\omega)$ of positions has no limit points. Let Ω be equipped with the weak topology, i.e., $\lim \omega_n = \omega$ means that $\lim x_i(\omega_n) = x_i(\omega)$ and $\lim v_i(\omega_n) = v_i(\omega)$ for each $i \in I$. This topology is a separable and metric one; the corresponding σ-algebra of Borel subsets of Ω will be denoted by $\mathscr{R}$.

The particle number and the total energy of a configuration ω in a y-centered disk of radius ρ are denoted by

$$N(\omega, y, \rho) = \sum_{i \in I} f_{y,\rho}(x_i) \tag{1}$$

and

$$H(\omega, y, \rho) = \frac{1}{2}\sum_{i\in I} f_{y,\rho}(x_i)\left[|v_i|^2 + \sum_{j\neq i} f_{y,\rho}(x_j)U(x_i - x_j)\right] \tag{2}$$

respectively, where $f_{y,\rho}(x) = 1$ if $|x - y| < \rho$, and $f_{y,\rho}(x) = 0$ otherwise. The quantity

$$\bar{H}(\omega) = \sup_{y\in Z^2}[g(|y|)]^{-2}H(\omega, y, g(|y|)) \tag{3}$$

is called the logarithmic energy fluctuation of ω; here $g(u) = 1 + \log(1 + u)$, where log denotes the natural logarithm. Let us remark that $\bar{H}$ is a lower semi-continuous function of ω.

The Markov time evolution will be constructed in the subset

$$\mathbf{\Omega}_0 = \{\omega; \bar{H}(\omega) < +\infty\} \tag{4}$$

of $\mathbf{\Omega}$; $\mathscr{R}_0 = \mathscr{R} \cap \mathbf{\Omega}_0$ denotes the σ-algebra of Borel subsets of $\mathbf{\Omega}_0$. Since either of (P) and (R) implies superstability of U, $\mathbf{\Omega}_0{}^q = \{\omega; \bar{H}(\omega) \leqslant q\}$ is a compact subset of $\mathbf{\Omega}_0$ for each $q < +\infty$.

Suppose now that we are given a sequence of independent, $\mathbf{R}^2$-valued standard Wiener processes $w_i = w_i(t)$, $t \geqslant 0$, $i \in I$, on a complete probability space $(\mathbf{C}, \mathscr{A}, \mathbf{P})$; the components of each w_i are uncorrelated, $w_i(0) = 0$. We may and do assume that the realizations of each w_i are continuous with probability one, e.g., $\mathbf{C}$ can be chosen as an infinite product of $\mathbf{C}[0, \infty)$ spaces. $\mathscr{A}_t$ denotes the σ-algebra generated by the family $w_i(s)$, $s \leqslant t$, $i \in I$, of random variables.

Now we are in a position to define what is a solution of (I). Let us remark that particles along a continuous trajectory ω_t in $\mathbf{\Omega}$ preserve their initial enumeration.

Definition. Consider a stochastic process $\boldsymbol{\omega}_t$, $t \geq 0$ on $(\mathbf{C}, \mathscr{A}, P)$ with state space $(\mathbf{\Omega}_0, \mathscr{R}_0)$, i.e., $\boldsymbol{\omega}_t = \omega_t(\mathbf{c})$ is a measurable mapping of $(\mathbf{C}, \mathscr{A}, \mathbf{P})$ into $(\mathbf{\Omega}_0, \mathscr{R}_0)$ for each $t \geqslant 0$. We say that $\boldsymbol{\omega}_t$ is a strong solution of (I) with initial configuration ω if $\omega_0 = \omega$, ω_t is $\mathscr{A}_t$-measurable for each $t \geqslant 0$; further, almost each trajectory of $\boldsymbol{\omega}_t$ is continuous and

$$\frac{d}{dt}x_i(\omega_t) = v_i(\omega_t)$$

$$v_i(\omega_t) = v_i(\omega_0) - \sum_{j\neq i}\int_0^t \operatorname{grad} U(x_i(\omega_s) - x_j(\omega_s))\,ds$$
$$- \lambda\int_0^t v_i(\omega_s)\,ds + \sigma w_i(t) \tag{I'}$$

hold for each $t \geqslant 0$, $i \in I$ along almost each trajectory $\omega_t(\mathbf{c})$ of $\boldsymbol{\omega}_t$. A solution

$\boldsymbol{\omega}_t$ is a tempered solution if $\bar{H}(\omega_t(\mathbf{c}))$ is bounded in finite intervals of time with probability one.

To avoid the possibility of misunderstanding, we have to clarify notation. ω_t is the value of the stochastic process $\boldsymbol{\omega}_t$ at time t; $\omega_t(\mathbf{c})$ is the trajectory of $\boldsymbol{\omega}_t$ corresponding to the random element $\mathbf{c} \in \mathbf{C}$. However, we do not indicate dependence of random variables on $\mathbf{c}$ in general; relations for ω_t as a function of time should be considered for almost each trajectory.

3. MAIN RESULT

Solutions will be constructed as a.s. weak limits of solutions to finite subsystems. Theorem 1 contains the basic results of Ref. 4 in the particular case of $\lambda = \sigma = 0$. Of course, the one-dimensional existence theorems of Ref. 3 also have similar, stochastic extensions.

Theorem 1. For each $\omega \in \mathbf{\Omega}_0$ there exists a tempered strong solution $\boldsymbol{\omega}_t = \varphi(t, \omega, \mathbf{c})$ of (I) such that $\omega_0 = \omega$ a.s., and this solution is unique in the sense that $\mathbf{P}(\omega_t(\mathbf{c}) = \bar{\omega}_t(\mathbf{c})$ for $t \geqslant 0) = 1$ whenever $\bar{\omega}_t$ is a tempered strong solution with $\bar{\omega}_0 = \omega$ a.s. The φ is jointly measurable in t, ω, $\mathbf{c}$, and it is a Markov process for each $\omega \in \mathbf{\Omega}_0$. Moreover, the restriction of $\varphi(t, \omega, \mathbf{c})$ to any of the subsets $\mathbf{\Omega}_0{}^q$ is a stochastically continuous function of $\omega \in \mathbf{\Omega}_0{}^q$; this continuity is uniform in finite intervals of time.

In contrast to the deterministic case of $\sigma = 0$, here (U) also is needed in the proof of existence. Without (U) only weak solutions can be constructed, i.e., ω_t is not necessarily $\mathscr{A}_t$-measurable. This measurability property is always needed when stochastic integrals are considered.

In view of Theorem 1, $\mathbf{P}_t = \mathbf{P}_t(\omega, A) = \mathbf{P}(\varphi(t, \omega, \mathbf{c}) \in A)$, $A \in \mathscr{R}_0$, is a semigroup of transition probabilities in $(\mathbf{\Omega}_0, \mathscr{R}_0)$, i.e., the translate $\mu_t = \mu\mathbf{P}_t$ of a probability measure μ on $(\mathbf{\Omega}_0, \mathscr{R}_0)$ is given by $\mu_t(A) = \int \mu(d\omega)\, \mathbf{P}_t(\omega, A)$. Let us remark that $\mathbf{\Omega}_0$ carries a wide class of probability measures defined originally on $(\mathbf{\Omega}, \mathscr{R})$. For example, if $\int \exp[pH(\omega, y, \rho)]\, \mu(d\omega) \leqslant \exp(q\rho^2)$ for $\rho \geqslant g(|y|)$ holds with some positive constants p and q, then the Borel–Cantelli lemma implies directly that $\mu(\mathbf{\Omega}_0) = 1$. This condition can be verified easily for such Gibbsian fields where the singularity of the interaction potential is not weaker than that of U; see Proposition 1 in Ref. 4.

The first problem arising here is certainly the description of time-invariant probability measures. A probability measure μ on $(\mathbf{\Omega}, \mathscr{R})$ is a canonical Gibbs state for U at temperature $T > 0$ if the particles are distributed in $\mathbf{R}^2$ according to a canonical Gibbsian point field with potential $(1/T)U$ (i.e., the field is specified by the conditional distributions of points in finite volumes V given the number of points in V and the configuration of points outside V; see Refs. 8 and 9), while velocities are completely independent of positions, and the velocity coordinates are identically distributed, independent Gaussian variables of zero means and variances T. Of course, $\mu(\mathbf{\Omega}_0) = 1$ in this case, too.

On the coincidence of canonical and grand canonical Gibbs states see Ref. 6.

Theorem 2. Let $\sigma > 0$ in (I); then a canonical Gibbs state μ for U at temperature $T > 0$ is time-invariant, i.e., $\mu = \mu\mathbf{P}_t$ if and only if $\lambda > 0$ and $T = \sigma^2/2\lambda$.

To indicate the dependence of the solutions on λ and σ, let $\varphi_{\lambda,\sigma}(t, \omega, \mathbf{c})$ denote the general solution of (I). The particular case $\lambda = \sigma = 0$ is of special interest; the classical solution $\varphi(t, \omega) = \varphi_{0,0}(t, \omega, \mathbf{c})$ has been constructed in Ref. 4.

Theorem 3. If λ and σ go to zero, then $\varphi_{\lambda,\sigma}(t, \omega, \mathbf{c})$ converges in probability to $\varphi(t, \omega)$.

It seems that the ergodic properties of the stochastic dynamics are nicer than those of the classical dynamics; such problems are to be discussed in a forthcoming paper.

4. A PRIORI PROBABILITY BOUND

In this section Proposition 2 of Ref. 4 will be extended to our stochastic situation; we prove a uniform bound for the distribution of $\bar{H}$ along solutions to finite subsystems of (I). Notation and methods follow those in Section 4 of Ref. 4.

Let us consider the motion of a finite number of particles within a potential barrier h; the external particles are frozen, $V \subset \mathbf{R}^2$ is a bounded, open set of smooth boundary; $h = h(x)$ is a nonnegative and twice continuously differentiable function if $x \in V$; $h(x) = 0$ if $x \notin V$; further, $\lim h(x) = +\infty$ when $x \in V$ tends to the boundary of V. Let $\omega \in \mathbf{\Omega}_0$, $J = J_V(\omega) = \{i \in I; x_i(\omega) \in V\}$ and define the random trajectory $\boldsymbol{\omega}_t = \varphi_V(t, \omega, \mathbf{c})$, $t \geqslant 0$, by $x_i(\omega_t) = x_i(\omega)$, $v_i(\omega_t) = 0$ if $i \notin J_V(\omega)$, while

$$\begin{aligned} dx_i(\omega_t) &= v_i(\omega_t)\,dt, \\ dv_i(\omega_t) &= -F_i(\omega_t)\,dt - \operatorname{grad} h(x_i(\omega_t))\,dt - \lambda v_i(\omega_t)\,dt + \sigma\,dw_i(t) \end{aligned} \qquad (\mathrm{J}_V)$$

if $i \in J_V(\omega)$ with initial condition $x_i(\omega_0) = x_i(\omega)$, $v_i(\omega_0) = v_i(\omega)$ for $i \in J_V(\omega)$; here

$$F_i(\bar{\omega}) = \sum_{j \neq i} \operatorname{grad} U(x_i(\bar{\omega}) - x_j(\bar{\omega})) \qquad (5)$$

i.e., the field of external particles is present, too. It is not quite trivial that (J_V) has a unique $\mathscr{A}_t$-measurable (i.e., strong) solution; only local existence and uniqueness follow from the finiteness of the total energy $H_V(\omega)$ by standard methods. For completeness we reproduce the argument of Exercise 5 in Section 4.5 of Ref. 7. Let $\tau = \tau(\mathbf{c})$ denote the random lifetime of the solution, $t \wedge \tau = \min(t, \tau)$, and observe that the stochastic differential of

$$H_V(\omega) = \frac{1}{2}\sum_{i\in J}\left[|v_i|^2 + 2h(x_i) + \sum_{j\notin J} U(x_i - x_j) + \sum_{j\neq i} U(x_i - x_j)\right]$$

along a solution of (J_V) is just

$$dH_V = -\lambda \sum_{i\in J} |v_i|^2\, dt + \sigma^2 |J|\, dt + \sigma \sum_{i\in J} v_i\, dw_i \tag{6}$$

where $|J|$ denotes the cardinality of $J = J_V(\omega)$. Thus from the Ito lemma [see (6)] we obtain that

$$H_V(\omega_{t\wedge\tau}) \leqslant H_V(\omega_0) + \sigma^2|J|t + \sigma\sum_{i\in J}\int_0^{t\wedge\tau} v_i\, dw_i \tag{7}$$

for each $t < +\infty$ with probability one. However, almost each trajectory of a stochastic integral has the following property: it is either bounded or oscillates between $-\infty$ and $+\infty$ in a finite interval of time; thus the lower boundedness of H_V results in $\lim_{t\to\tau} H_V(\omega_{t\wedge\tau}) < +\infty$. Therefore the local existence theorem implies that $\mathbf{P}(\tau < +\infty) = 0$, i.e., ($\mathrm{J}_V$) has a unique global solution, and $\varphi_V(t, \omega, \mathbf{c})$ is an $\mathscr{A}_t$-measurable Markov process.

The first step of the proof of Theorem 1 is to extend the stochastic version (6) of the law of energy conservation to infinite systems. For this we introduce a nonnegative and additive modification W of the total energy. Let $f = f(u)$ denote such a twice continuously differentiable nondecreasing function that:

(i) $f(u) = 0$ if $u \leqslant -3R$; $f(u) = 1$ if $u \geqslant 0$; $f(-5R/2) = 1/9$; $f(-R/2) = 8/9$.

(ii) f is convex for $u \leqslant -R/2$ and it is concave if $u \geqslant -5R/2$, i.e., f is linear if $-5R/2 \leqslant u \leqslant -R/2$.

(iii) There exists a constant d_1 such that $|f'(u)|^2 \leqslant d_1 f(u)$

If $\omega \in \mathbf{\Omega}$, $y \in \mathbf{R}^2$, $\rho > 0$, then W is defined as

$$W(\omega, y, \rho) = \sum_{i\in I} f(\rho - |x_i - y|) W_i(\omega) \tag{8}$$

where

$$W_i(\omega) = 1 + |v_i|^2 + 2h(x_i) + \sum_{j\neq i} \delta_R f(3R - 3|x_i - x_j|) + \sum_{j\neq i} U(x_i - x_j)$$

and $\delta_R = a/b$ if (R) holds, $\delta_R = 0$ otherwise. Let us remark that $W_i(\omega) \geqslant 1$ in view of (E) and W is a nondecreasing function of ρ. The logarithmic fluctuation of W is defined as

$$\overline{W}(\omega) = \sup_{y\in\mathbf{Z}^2}[g(|y|)]^{-2}W(\omega, y, g(|y|)) \tag{9}$$

Some basic properties of $\overline{W}$ are summarized in the following lemma:

Lemma 1. There exist constants a_1, b_1, c_1 depending only on U such that

$$W(\omega, x, \rho) \leqslant a_1\rho^2\overline{W}(\omega) \tag{10}$$

whenever $x \in \mathbf{R}^2$, $\rho \geqslant g(|x|)$, and further,

$$\overline{H}(\omega) \leqslant \overline{W}(\omega) \leqslant b_1 + c_1\overline{H}(\omega) + 2\sum_{i\in I} h(x_i(\omega)) \tag{11}$$

$$|v_i(\omega)| \leqslant a_1 g(|y| + \rho)|\overline{W}(\omega)|^{1/2} \tag{12}$$

if $|x_i(\omega) - y| \leqslant \rho + 5R$, and

$$N(\omega, x_i(\omega), 2R) \leqslant 1 + a_1 g(|y| + \rho)|\overline{W}(\omega)|^{1/2} \tag{13}$$

if $|x_i(\omega) - y| \leqslant \rho + 5R$.

Proof. Let D_y denote the disk of center $y \in \mathbf{Z}^2$ and radius $g(|y|)$; first we show that there exists such a subset $\mathbf{Z}_0{}^2$ of $\mathbf{Z}^2$ that only a fixed number of disks D_y, $y \in \mathbf{Z}_0{}^2$, can have a nonempty intersection, and the union of all disks D_y, $y \in \mathbf{Z}_0{}^2$, covers $\mathbf{R}^2$. For this set $r_1 = 1$, and $r_{k+1} = r_k + g(r_k)$ for $k \in I$; let n_k denote the smallest integer larger than $8r_k/g(r_k)$. For each $k \in I$ we choose n_k points from the origin-centered circle of radius r_k in such a way that they form a regular polygon; $\mathbf{R}_0{}^2$ consists of the above described points, and $\mathbf{Z}_0{}^2$ is the set of such $y \in \mathbf{Z}^2$ that $|x - y| \leqslant 2$ for some $x \in \mathbf{R}_0{}^2$. Since $\lim[g(r_{k+1}) - g(r_k)] = 0$, it follows easily that $\mathbf{Z}_0{}^2$ has the property we need; thus

$$\begin{aligned} W(\omega, x, \rho) &\leqslant \sum W(\omega, y, g(|y|)) \leqslant \sum \overline{W}(\omega)g^2(|y|) \\ &\leqslant \overline{W}(\omega)[g(|x| + \rho + 3R + 2) + \rho + 3R + 2]^2 n_0 \end{aligned} \tag{14}$$

where both sums are over such $y \in \mathbf{Z}_0{}^2$ that $|x - y| \leqslant \rho + 3R + 2$; n_0 is the maximal multiplicity of the covering $\{D_y; y \in \mathbf{Z}_0{}^2\}$ of $\mathbf{R}^2$. Since $\rho \geqslant g(|x|)$ in (14), the subadditivity of g implies (10) directly.

Condition (11) follows from the superstability of U in a similar way as (10) has been deduced; see Ref. 8 and the proof of Proposition 4 in Ref. 4; (12) and (13) are obviously true.

Now we turn to the problem of time evolution. Let $\boldsymbol{\omega}_t$ denote either a tempered strong solution of (I) or $\boldsymbol{\omega}_t = \varphi_V(t, \omega, \mathbf{c})$ for (J_V); $h = 0$ in the definition of W in the first case. We define $W'(\bar{\omega}, y, \rho)$ as the time derivative of W at $\bar{\omega}$ along the classical solution $\boldsymbol{\omega}_t{}^0$, i.e.,

$$W'(\omega_t{}^0, y, \rho) = (d/dt)W(\omega_t{}^0, y, \rho),$$

where $\boldsymbol{\omega}_t^0$ is the solution of (I) or (J_V) with $\lambda = \sigma = 0$, respectively. The explicit expression of W' is given by the corresponding Poisson bracket. In view of the basic estimate (6) of Ref. 4, there exists a constant K_0 depending only on U such that

$$W'(\bar{\omega}, y, \rho) \leqslant K_0 g(|y| + \rho)[\overline{W}(\bar{\omega})]^{1/2} \frac{\partial}{\partial \rho} W(\bar{\omega}, y, \rho) + K_0 W(\bar{\omega}, y, \rho) \quad (15)$$

holds for each $\bar{\omega} = \omega_t$, $y \in \mathbf{R}^2$, and $\rho > 0$; since $dx_i = v_i\, dt$ even if $x_i \notin V$, the presence of the external field and of frozen particles does not involve any change in the proof of (15) in comparison with that of (6) in Ref. 4.

For each $k \in I$ and $y \in \mathbf{Z}^2$ we define a stochastic process $\rho_k(t)$, $t \geqslant 0$, as the a.s. unique solution of the integral equation

$$\rho_k(t) = kg(|y|) - K_0 \int_0^t g(|y| + |\rho_k(s)|) z'(s)\, ds \quad (16)$$

where

$$z(t) = \int_0^t [\overline{W}(\omega_s)]^{1/2}\, ds$$

It is easy to check that $\rho_k(t)$ is $\mathscr{A}_t$-measurable; the trajectories of ρ_k are differentiable and decreasing, $\rho_{k+1}(t) - \rho_k(t) \leqslant g(|y|)$ a.s. for each $t \geqslant 0$; further, $\tau_k = \sup\{t \geqslant 0;\ \rho_k(t) \geqslant g(|y|)\}$ is a Markov time with respect to $\mathscr{A}_t$ such that $\tau_k < \tau_{k+1} < +\infty$ and $\lim \tau_k = +\infty$ a.s. Put $K = K_0 + \sigma^2$; in view of the Ito lemma the stochastic differential of $e^{-Kt} W(\omega_t, y, \rho_k(t))$ is

$$\begin{aligned} d(e^{-Kt}W) = e^{-Kt}\left[-KW + W' + \left(\frac{\partial}{\partial \rho} W\right)\rho_k'\right] dt \\ - \lambda e^{-Kt} \sum_{i \in J} f_i |v_i|^2\, dt + \sigma^2 e^{-Kt} \sum_{i \in J} f_i\, dt + \sigma e^{-Kt} \sum_{i \in J} f_i v_i\, dw_i(t) \quad (17) \end{aligned}$$

where $f_i = f(\rho_k(t) - |y - x_i(\omega_t)|)$, and $J = J_V(\omega)$ if $\boldsymbol{\omega}_t = \varphi_V(t, \omega, \mathbf{c})$ and $J = I$ if $\boldsymbol{\omega}_t$ is a tempered solution of (I). Since the sums in (17) are finite in the sense that $f_i = 0$ apart from a finite number of particles, a straightforward approximation procedure shows that (17) remains in force even in the second case. We have to remark that among the stochastic variables $\rho_k(t)$, $x_i(t)$, $v_i(t)$ only the v_i have a proper stochastic differential; thus the twice continuous differentiability of W is needed only in v_i. Therefore (17) certainly holds whenever $t \leqslant \tau_k$.

Lemma 2. For each $k \in I$, $y \in \mathbf{Z}^2$, and $u > 0$ we have

$$\sup_{t \geqslant 0} e^{-Kt} W(\omega_{t \wedge \tau_k}, y, \rho_k(t \wedge \tau_k)) \leqslant W(\omega_0, y, g(|y|)k) + u$$

with a probability larger than $1 - e^{-2u}$.

Proof. Using (15), (16), $\lambda \geqslant 0$, and $1 + f_i|v_i|^2 \leqslant W_i(\omega_t)$ [see (8)], we find that (17) becomes

$$\{\exp[-K(t \wedge \tau_k)]\} W(\omega_{t\wedge\tau_k}, y, \rho_k(t \wedge \tau_k)) \\ \leqslant W(\omega_0, y, g(|y|)k) + \sum_{i\in J}\left[\int_0^{t\wedge\tau_k} p_i(s)\, dw_i(s) - \int_0^{t\vee\tau_k} |p_i(s)|^2\, ds\right]$$

where $p_i = \sigma e^{-Ks} f_i v_i$. Thus the exponential supermartingale inequality [see (6) in Section 2.3 and Exercise 5 in Section 2.9 of Ref. 7]

$$\mathbf{P}\left[\sup_{t\geqslant 0}\left(\int_0^t \sum_{i\in J} \bar{p}_i\, dw_i - \int_0^t \sum_{i\in J} |\bar{p}_i|^2\, ds\right) > u\right] \leqslant e^{-2u}$$

with $\bar{p}_i(s) = p_i(s)$ if $s < \tau_k$, $\bar{p}_i = 0$ otherwise, implies the statement of the lemma. To verify the above inequality in the case of $J = I$, again a standard approximation procedure is needed.

Observe first that

$$\sum_{y\in\mathbf{Z}^2} \sum_{k\in I} \exp[-4kg^2(|y|)] < 1$$

Therefore, replacing u by $u + 2kg^2(|y|)$ in Lemma 2, and using also (10), we obtain that

$$\sup_{t\geqslant 0} e^{-Kt} W(\omega_{t\wedge\tau_k}, y, \rho_k(t \wedge \tau_k)) \leqslant a_1 \overline{W}(\omega_0) k^2 g^2(|y|) + 2kg^2(|y|) + u \tag{18}$$

holds simultaneously for each $k \in I$ and $y \in \mathbf{Z}^2$ with a probability larger than $1 - e^{-u}$. Define now $k = k_t$, $t \geqslant 0$, as the smallest integer $k \in I$ for which $\rho_k(t) > g(|y|)$; then $\tau_k > t$ and $\rho_k(t) \leqslant 2g(|y|)$ as $\rho_{k-1}(t) \leqslant g(|y|)$; thus, choosing k as $k = k_t$ in (18), it follows that

$$e^{-Kt} \overline{W}(\omega_t) \leqslant a_1 \overline{W}(\omega_0) k_t^2 + 2k_t + u \tag{19}$$

for each $t \geqslant 0$ with probability at least $1 - e^{-u}$. On the other hand

$$2g(|y|) \geqslant k_t g(|y|) - K\int_0^t g(|y| + |\rho_{k_t}(s)|) z'(s)\, ds$$

whence

$$k_t \leqslant 2 + Kz(t)[1 + g(k_t)] \leqslant 2 + Kz(t)(2 + 2\sqrt{k_t}) \tag{20}$$

follows by a direct calculation; thus $\sqrt{k_t} \leqslant 2 + 4Kz(t)$. Substituting the last inequality into the first part of (20), we obtain that

$$k_t \leqslant 2 + Kz(t)(1 + g\{[2 + 4Kz(t)]^2\}) \tag{21}$$

Relations (19) and (21) are summarized in the following lemma:

Lemma 3. Let $u \geqslant 1$ and $w \geqslant 1$ and suppose that $\overline{W}(\omega_0) \leqslant w$, where

$\boldsymbol{\omega}_t$, $t \geqslant 0$, is either a tempered strong solution of (I), or $\boldsymbol{\omega}_t = \varphi_V(t, \omega, \mathbf{c})$ for (J_V). Then there exists a constant M depending only on U such that

$$\mathbf{P}\left(\sup_{t \geqslant 0}\{M^{-1}e^{-Mt}z'(t) - w[1 + z(t)g(z(t))]\} \leqslant u\right) \geqslant 1 - e^{-u}$$

the process $z(t)$ has been defined in (16).

Proof. It is immediate; for notational convenience $\sqrt{w}$ and $\sqrt{u}$ have been estimated by w and u, respectively.

Now we are in a position to prove the basic probability estimate for $\overline{W}(\omega_t)$.

Proposition 1. For each $w \geqslant 1$ there exists a continuous function $q_w(t)$, $t \geqslant 0$, such that

$$\mathbf{P}\left\{\sup_{0 \leqslant s \leqslant t} \overline{W}(\omega_s) > \exp[q_w(t)g(u)]\right\} \leqslant e^{-u}$$

for each $u \geqslant 1$, $t \geqslant 0$, whenever $\overline{W}(\omega_0) \leqslant w$; $\boldsymbol{\omega}_t$ is the same as in Lemma 3; it is defined before (15).

Proof. Define $z_u = z_u(t)$ as the solution of the differential equation

$$z' = Me^{Mt}[w + wzg(z) + u] \tag{22}$$

with initial condition $z_u(0) = 0$; then in view of Lemma 3 we have

$$\mathbf{P}\left[\sup_{0 \leqslant s \leqslant t} |\overline{W}(\omega_s)|^{1/2} > z_u'(t)\right] \leqslant e^{-u}$$

It is easy to check that $z_u(t) < +\infty$ for each $t \geqslant 0$. Therefore it is enough to show that $z_u(t) \leqslant r_u(t)$ for $t \geqslant 0$, where $r_u(t) = \exp[z_{11}(t)g(u)] - u$, and z_{11} is the solution of (22) for $u = 1$ with initial condition $z_{11}(0) = 1$. Observe that $z_{11}g(u) = \log(u + r_u)$ in the time derivative

$$r_u'(t) = Me^{Mt}[w + wz_{11}g(z_{11}) + 1](r_u + u)g(u)$$

of r_u; further, u, w, $g(u)$, $g(z_{11})$ are not less than 1; consequently,

$$r_u'(t) > Me^{Mt}[w + r_u g(r_u) + u] \geqslant z_u'(t)$$

whenever $z_u(t) \leqslant r_u(t)$. Since $z_u(0) < r_u(0)$, this is possible only if $z_u \leqslant r_u$ for each $t \geqslant 0$, which proves the statement of Proposition 1.

The essential content of Proposition 1 is the weak compactness of the set of probability measures for solutions $\boldsymbol{\omega}_t$ of (I) or (J_V) such that $\overline{W}(\omega_0) \leqslant w$.

5. PROOF OF THEOREM 1

We show that there exists the a.s. limit $\varphi(t, \omega, \mathbf{c}) = \lim \varphi_V(t, \omega, \mathbf{c})$ as V tends to $\mathbf{R}^2$, and φ is the unique tempered strong solution. The proof is based on the following quasi-Lipschitz property of the right-hand side of (I) and (J_V): in view of (U) we have

$$|F_i(\omega) - F_i(\bar{\omega})| \leqslant L_1\left[|J_i'|\,|x_i(\omega) - x_i(\bar{\omega}) + \sum_{j\in J_i'} |x_j(\omega) - x_j(\bar{\omega})|\right] \quad (23)$$

where

$$L_1 = L[\max\{g^2(|x_i(\omega)|)\bar{W}(\omega), g^2(|x_i(\bar{\omega})|)\bar{W}(\bar{\omega})\}]^d$$
$$J_i' = \{i \in I; \min\{|x_i(\omega) - x_j(\omega)|, |x_i(\bar{\omega}) - x_j(\bar{\omega})|\} \leqslant R\}$$

The cardinality $|J_i'|$ of J_i' can be estimated via (13).

The external field $h = h_V$ in (J_V) is almost arbitrary; we assume that $h_V(x) = 0$ even if the distance of x from the boundary of V is larger than R. Consider now the time evolution of

$$d(\omega, \bar{\omega}, y, \rho) = \sum_{i\in I} f_i \bar{f}_i(|x_i - \bar{x}_i|^2 + |v_i - \bar{v}_i|^2) \quad (24)$$

along two solutions $\boldsymbol{\omega}_t$ and $\bar{\boldsymbol{\omega}}_t$ of (I) or (J_V); here and in what follows the usual abbreviations $x_i = x_i(\omega)$, $v_i = v_i(\omega)$, $\bar{x}_i = x_i(\bar{\omega})$, $\bar{v}_i = v_i(\bar{\omega})$ are used; f_i, f_i' and $\bar{f}_i, \bar{f}_i'$ denote the value and the derivative of f at $\rho - |x_i - y|$ and at $\rho - |\bar{x}_i - y|$, respectively; $D_y(\rho)$ is the disk of center y and radius $\rho > 0$.

Let $\boldsymbol{\omega}_t = \varphi_V(t, \omega, \mathbf{c})$, $\bar{\boldsymbol{\omega}}_t = \varphi_{\bar{V}}(t, \omega, \mathbf{c})$; then x_i, $\bar{x}_i$, and $v_i - \bar{v}_i$ are differentiable functions of time; thus

$$\begin{aligned}(d/dt)&d(\omega_t, \bar{\omega}_t, y, \rho)\\ &\leqslant \sum_{i\in I} (f_i'\bar{f}_i|v_i| + f_i\bar{f}_i'|\bar{v}_i|)(|x_i - \bar{x}_i|^2 + |v_i - \bar{v}_i|^2)\\ &\quad + 2\sum_{i\in I} f_i\bar{f}_i|v_i - \bar{v}_i|[|x_i - \bar{x}_i| + \lambda|v_i - \bar{v}_i| + |F_i(\omega_t) - F_i(\bar{\omega}_t)|]\end{aligned} \quad (25)$$

provided that $D_y(\rho + 4R) \subset V \cap \bar{V}$. Observe that f' is a bounded function; further, $2|v_i - \bar{v}_i|\,|x_j - \bar{x}_j| \leqslant |x_j - \bar{x}_j|^2 + |v_i - \bar{v}_i|^2$; thus d' can be estimated by $d(\omega_t, \omega_t, y, \rho + 4R)$ as follows. Substituting (23) into (25) and estimating $|v_i|$, $|\bar{v}_i|$, and $|J_i'|$ via (12) and (13), we obtain that there exists a constant L_2 depending only on U such that

$$(d/dt)d(\omega_t, \bar{\omega}_t, y, \rho) \leqslant QM_1 d(\omega_t, \bar{\omega}_t, y, \rho + 4R) \quad (26)$$

where the sequence $M_n = M_n(y, \rho)$ is defined by the recursive formulas $M_0(y, \rho) = 1$, $M_1(y, \rho) = g^{2d+1}(|y| + \rho)$, and

$$M_n(y, \rho) = (1/n)M_{n-1}(y, \rho)M_1(y, \rho + 4Rn - 4R)$$

if $n > 1$; further

$$Q = Q(t, \omega_0, \bar{\omega}_0) = \sup_{s\leqslant t} L_2[\max\{1, \bar{W}(\omega_s), \bar{W}(\bar{\omega}_s)\}]^{d+1/2} \quad (27)$$

Therefore

$$\sup_{s\leqslant t} d(\omega_s, \bar\omega_s, y, \rho)$$
$$\leqslant d(\omega_0, \bar\omega_0, y, \rho) + Q(t, \omega_0, \bar\omega_0)M_1(y, \rho)\int_0^t d(\omega_s, \bar\omega_s, y, \rho + 4R)\, ds \quad (28)$$

holds for each $t \geqslant 0$ with probability one, provided that $D_y(\rho + 4R) \subset V \cap \bar V$. Let us remark that (28) remains in force even if $\boldsymbol{\omega}_t$ and $\boldsymbol{\omega}_t$ are tempered strong solutions of (I); in this case no restriction is needed on y and ρ. Iterating (28) as many times as possible, the basic tool of this section is obtained.

Lemma 4. Let $\boldsymbol{\omega}_t$ and $\boldsymbol{\omega}_t$ denote either tempered strong solutions of (I), or $\boldsymbol{\omega}_t = \varphi_V(t, \omega, \mathbf{c})$, $\bar{\boldsymbol{\omega}}_t = \varphi_V(t, \omega, \mathbf{c})$ with $D_y(\rho + 4Rn) \subset V \cap \bar V$. Then there exists a constant L_3 depending only on U such that

$$\sup_{s\leqslant t} d(\omega_s, \bar\omega_s, y, \rho)$$
$$\leqslant L_3 M_n(y, \rho) Q^{n+2}(t, \omega_0, \bar\omega_0) n^4 t^n$$
$$+ \sum_{k=0}^{n-1} [tQ(t, \omega_0, \bar\omega_0)]^k M_k(y, \rho) d(\omega_0, \bar\omega_0, y, \rho + 4kR)$$

holds with probability one for each $t \geqslant 0$ and $y \in \mathbf{R}^2$, $\rho > 0$ satisfying $\rho \leqslant 4Rn$ and $\rho + 4Rn \geqslant g(|y|)$.

Proof. Iterating (28) $n - 1$ times, we obtain that

$$\sup_{s\leqslant t} d(\omega_s, \bar\omega_s, y, \rho) \leqslant t^n Q^n M_n \sup_{s\leq t} d(\omega_s, \bar\omega_s, y, \rho + 4Rn)$$
$$+ \sum_{k=0}^{n-1} t^k Q^k M_k d(\omega_0, \bar\omega_0, y, \rho + 4kR) \quad (29)$$

On the other hand, using $|x_i - \bar x_i| \leqslant 4(\rho + 3R)$ and

$$|v_i - \bar v_i|^2 \leqslant 2(|v_i|^2 + |\bar v_i|^2)$$

in (24), it follows by Lemma 1 that

$$d(\omega, \bar\omega, y, \rho) \leqslant L_4 \rho^4 \max\{\bar W(\omega), \bar W(\bar\omega)\} \quad (30)$$

whenever $\rho \geqslant g(|y|)$; L_4 is a universal constant. A comparison of (27), (29), and (30) results in the statement of the lemma.

We can consider $d(\omega_t, \bar\omega_t, y, \rho)$ as a reasonable measure for the deviation of solutions from each other only if we a priori know that the particles are localizable, i.e., $x_i(\omega_t)$ remains in a controllable neighborhood of $x_i(\omega_0)$.

Proposition 2. Let $\boldsymbol{\omega}_t$ denote either a tempered strong solution of (I) or $\boldsymbol{\omega}_t = \varphi_V(t, \omega, \mathbf{c})$, and suppose that $\bar W(\omega_0) \leqslant w$, $w \geqslant 1$. Then for each

$t \geqslant 0$, $y \in \mathbf{R}^2$, there exists a positive $\epsilon = \epsilon(t, y, w)$ such that $\rho \geqslant 4r > 0$ implies that

$$\mathbf{P}\left[\sup_{i \in J(y,r)} \sup_{s \leqslant t} |x_i(\omega_s) - y| > \rho\right] \leqslant \exp(1 - \epsilon\rho^{1/q})$$

where $J(y, r) = \{i \in I; |x_i(\omega_0) - y| \leqslant r\}$, and $q = q_w(t)$ is the same as in Proposition 1.

Proof. Let S denote the minimal radius such that $x_i(\omega_s) \in D_y(S)$ if $s \leqslant t$, $i \in J(y, r)$. Since $|x_i(\omega_s)| \leqslant |y| + S$ and

$$|x_i(\omega_s) - y| \leqslant r + |x_i(\omega_s) - x_i(\omega_0)| \leqslant r + \int_0^t |v_i(\omega_s)|\, ds$$

in this case, in view of (12) we have

$$S \leqslant r + a_1 g(|y| + S) z(t) \tag{31}$$

where $z(t)$ has been defined in (16). Using the subadditive property of g and $g(S) \leqslant 1 + \sqrt{S}$, it follows that

$$S \leqslant r + a_1[1 + g(|y|) + \sqrt{S}]z(t)$$

Thus

$$\sqrt{S} \leqslant \sqrt{r} + L_5 z(t)$$

provided that $r \leqslant S$; L_5 is a new constant depending on y. Hence

$$S \leqslant 2r + 2L_5^2 z^2(t) \leqslant \tfrac{1}{2}\rho + 2(tL_5)^2 \sup_{s \leqslant t} \overline{W}(\omega_s) \tag{32}$$

follows directly, and (32) holds even if $S < r$. Observe now that $\epsilon = \epsilon(t, y, w) > 0$ can be chosen to be so small that

$$2(tL_5)^2 \exp[q_w(t) g(u)] = 2(tL_5)^2 e^q (1 + u)^q \leqslant \tfrac{1}{2}\rho$$

holds for $u = \epsilon\rho^{1/q} - 1$; $q = q_w(t)$. This means that $P(S > \rho) \leqslant e^{1-u}$ in view of Proposition 1, which proves Proposition 2.

Now we prove the existence of limiting solutions. Remember that the weak topology of $\mathbf{\Omega}_0$ is defined in the following way. For each finite $J \subset I$ set

$$|\omega - \bar{\omega}|_J = \left\{\sum_{i \in J} [|x_i(\omega) - x_i(\bar{\omega})|^2 + |v_i(\omega) - v_i(\bar{\omega})|^2]\right\}^{1/2} \tag{33}$$

Then $\lim \omega_n = \omega$ means that $\lim|\omega - \omega_n|_J = 0$ for each finite $J \subset I$. Due to Proposition 2, $|\omega_t - \bar{\omega}_t|_{J(y,r)}$ can be estimated by $[d(\omega_t, \bar{\omega}_t, y, \rho)]^{1/2}$ with a probability close to one if ρ is large enough.

Let $\omega \in \mathbf{\Omega}_0$, $V_n = D_0(8Rn + R)$, and $\boldsymbol{\omega}_t^n = \varphi_{V_n}(t, \omega, \mathbf{c})$; we may assume that the external field $h = h_n$ in (J_{V_n}) has been chosen in such a way that sup $\overline{W}(\omega_0{}^n) \leqslant w < +\infty$ and $w \geqslant 1$. We want to apply Lemma 4 and Proposition 2 to $\boldsymbol{\omega}_t = \boldsymbol{\omega}_t^{n+m}$ and $\overline{\boldsymbol{\omega}}_t = \boldsymbol{\omega}_t^{n+m+1}$, where $m \in I$ is fixed, $r = Rm$, and $y = 0$. Proposition 1 implies

$$\mathbf{P}\{Q > L_2 \exp[(d + 1/2)q_w(t)g(u)]\} \leqslant 2e^{-u}$$

for $Q = Q(t, \omega_0, \overline{\omega}_0)$, and the simultaneous bound of Proposition 2 for $\boldsymbol{\omega}_t$ and $\overline{\boldsymbol{\omega}}_t$ is $2 \exp(1 - \epsilon\rho^{1/q})$, with $q = q_w(t)$ and $\epsilon = \epsilon(t, 0, w)$. Put $\rho = 4Rn$ in Proposition 2 and in Lemma 4, and $u = 2g(n)$ in Proposition 1; since $d(\omega_0^{n+m}, \omega_0^{n+m+1}, 0, \rho + 4kR) = 0$ in Lemma 4 as $k \leqslant n$, we obtain that

$$\mathbf{P}\left\{\sup_{s \leqslant t}|\omega_s^{n+m} - \omega_s^{n+m+1}|_{J(r)} > \delta_n(t, w)\right\} \leqslant 2\exp[-2g(n)] + 2\exp[1 - \epsilon(4Rn)^{1/q}] \tag{34}$$

where $J(r) = \{i \in I; |x_i(\omega)| \leqslant r\}$ and

$$\delta_n(t, w) = \{L_3 L_2^{n+2} n^4 t^n M_n(0, 4Rn) \exp[q(n + 2)(d + 1/2)g(2g(n))]\}^{1/2}$$

It is easy to check that $\sum \delta_n(t, w) < +\infty$ and the sum over n of terms on the right-hand side of (34) is finite, too; therefore the Borel–Cantelli lemma implies

$$\mathbf{P}\left[\sup_{s \leqslant t} \sum_{n=1}^{\infty} |\omega_s{}^n - \omega_s^{n+1}|_{J(r)} < +\infty\right] = 1 \tag{35}$$

for each $t > 0$, $r = Rm$ if $m \in I$, which proves that there exists the limit $\varphi(t, \omega, \mathbf{c}) = \lim \varphi_{V_n}(t, \omega, \mathbf{c})$ with probability one for each $\omega \in \mathbf{\Omega}_0$.

We have to verify that $\varphi(t, \omega, \mathbf{c})$ is a tempered strong solution. Since $\overline{W}$ is a lower semicontinuous function of $\omega \in \mathbf{\Omega}_0$, we have $\overline{W}(\varphi(t, \omega, \mathbf{c})) \leqslant \liminf \overline{W}(\varphi_{V_n}(t, \omega, \mathbf{c}))$ a.s.; further, $\overline{W} \geqslant 0$; thus the Fatou–Lebesgue theorem and Proposition 1 imply that

$$\int_{\mathbf{C}} \sup \overline{W}(\varphi(s, \omega, \mathbf{c}))\,\mathbf{P}(d\mathbf{c}) \leqslant \liminf \int_{\mathbf{C}} \sup_{s \leqslant t} \overline{W}(\varphi_{V_n}(s, \omega, \mathbf{c}))\,\mathbf{P}(d\mathbf{c}) \leqslant p_w(t) \tag{36}$$

if $\overline{W}(\omega) \leqslant w$, where $p_w(t)$ is a continuous function of $t \geqslant 0$ for each $w \leqslant +\infty$. Hence

$$\mathbf{P}\left[\sup_{s \leqslant t} \overline{W}(\varphi(s, \omega, \mathbf{c})) < +\infty\right] = 1 \tag{37}$$

Thus the interparticle distances in $\omega_t = \varphi(t, \omega, \mathbf{c})$ are positive with probability one; consequently $\boldsymbol{\omega}_t$ satisfies (I′). The measurability properties of φ are direct consequences of those of φ_{V_n}; as a solution of (I′), φ is automatically a Markov process, while temperedness of φ is just (37). Uniqueness of tempered strong solutions and their continuous dependence on initial data are a direct consequence of the following result.

Proposition 3. Suppose that $\boldsymbol{\omega}_t$ and $\bar{\boldsymbol{\omega}}_t$ are tempered strong solutions of (I) with $\max\{\overline{W}(\omega_0), \overline{W}(\bar{\omega}_0)\} \leqslant w$, $w \geqslant 1$. Then for each sequence $u_k \geqslant 1$, $t > 0$, $y \in \mathbf{R}^2$, $\rho \geqslant 4r > 0$ we have

$$\mathbf{P}\left[\sup_{s \leqslant t} |\omega_s - \bar{\omega}_s|_{J(y,r)} > D(t, \omega_0, \bar{\omega}_0, y, \rho, w, u_k)\right]$$
$$\leqslant 2\exp(1 - \epsilon\rho^{1/q}) + 2\sum_{k=0}^{\infty} \exp(-u_k)$$

where $J(y, r) = \{i \in I; \min\{|x_i(\omega_0) - y|, |x_i(\bar{\omega}_0) - y|\} \leqslant r\}$, $q = q_w(t)$ and $\epsilon = \epsilon(t, y, w)$ are the same as in Propositions 1 and 2, and

$$D^2(t, \omega_0, \bar{\omega}_0, y, \rho, w, u_k)$$
$$= \sum_{k=0}^{\infty} (L_2 t)^k \exp[qk(d + 1/2)g(u_k)]M_k(y, \rho)d(\omega_0, \bar{\omega}_0, y, \rho + 4kR)$$

Further, D is finite, e.g., if u_k increases as fast as a power of $g(k)$, and in such cases $\lim \omega_0{}^n = \omega$ implies $\lim_n D(t, \omega_0, \omega_0{}^n, y, \rho, w, u_k) = 0$, provided that $\overline{W}(\omega_0{}^n) \leqslant w$.

Proof. Let n go to infinity in Lemma 4; then the statement follows from Propositions 1 and 2 in the same way as (34) has been obtained.

As a direct consequence of Proposition 2, we obtain that $\lim \omega_0{}^n = \omega_0$ implies

$$\lim_n \mathbf{P}\left(\sup_{s \leqslant t} |\omega_s{}^n - \omega_s|_{J(y,r)} > D\right) = 0 \tag{38}$$

for each $t > 0$, $D > 0$, $y \in \mathbf{R}^2$, and $r > 0$, which is a stronger statement than that of Theorem 1 about continuous dependence on initial data.

6. PROOF OF THEOREM 2

First we show that the canonical Gibbs distributions at temperature $T = \sigma^2/2\lambda$ are invariant under the Markov time evolution defined by (J_V). Let ω_V and $\omega_V{}^c$ denote the configuration inside V and that outside V, respectively, i.e., $\omega = (\omega_V, \omega_V{}^c)$. We may assume that the particles inside V are numbered by $1, 2, \ldots, n$; the configuration $\omega_V{}^c$ of frozen particles and the number $n = |J_V(\omega)|$ of moving particles are fixed. Then $f_V(\omega_V|\omega_V{}^c) = Z^{-1}\exp[-(1/T)H_V(\omega)]$ is the density of the canonical Gibbs distribution in V with respect to the $4n$-dimensional Lebesgue measure; $T > 0$ is the temperature; the total energy $H_V = H_V(\omega)$ is explicitly defined before (6).

For notational convenience set $x_i = (q_{2i-1}, q_{2i})$ and $v_i = (p_{2i-1}, p_{2i})$ if $i = 1, 2, \ldots, n$, and let $\mathbf{P}_i = \partial/\partial p_i$ and $\mathbf{Q}_i = \partial/\partial q_i$ denote the operators of differentiating functions of ω_V with respect to p_i and q_i, respectively. Then the generator $\mathbf{G}$ of the Markov process φ_V defined by (J_V) can be written as

$$\mathbf{G} = \sum_{i=1}^{2n} [\tfrac{1}{2}\sigma^2 \mathbf{P}_i^2 - \lambda p_i \mathbf{P}_i - (\mathbf{Q}_i H_V)\mathbf{P}_i + p_i \mathbf{Q}_i] \tag{39}$$

and the formal adjoint $\mathbf{G}^*$ of the differential operator $\mathbf{G}$ is acting as

$$\begin{aligned} \mathbf{G}^* f &= \sum_{i=1}^{2n} \{\tfrac{1}{2}\sigma^2 \mathbf{P}_i^2 f + \lambda \mathbf{P}_i(p_i f) + \mathbf{P}_i[(\mathbf{Q}_i H_V) f] - \mathbf{Q}_i(p_i f)\} \\ &= \sum_{i=1}^{2n} [\tfrac{1}{2}\sigma^2 \mathbf{P}_i^2 f + \lambda f + \lambda p_i \mathbf{P}_i f + (\mathbf{Q}_i H_V)(\mathbf{P}_i f) - p_i \mathbf{Q}_i f] \end{aligned} \tag{40}$$

Since $\mathbf{P}_i f_V = -T^{-1} p_i f_V$, $\mathbf{P}_i^2 f_V = -T^{-1} f_V + T^{-2} p_i^2 f_V$, and

$$\mathbf{Q}_i f_V = -T^{-1}(\mathbf{Q}_i H_V) f_V,$$

we have

$$\mathbf{G}^* f_V = 2n\left(\lambda - \frac{\sigma^2}{2T}\right) f_V - T^{-1}\left(\lambda - \frac{\sigma^2}{2T}\right) f_V \sum_{i=1}^{2n} p_i^2$$

so that f_V satisfies the stationary Kolmogorov–Fokker–Planck equation $\mathbf{G}^* f = 0$ if and only if $T = \sigma^2/2\lambda$.

Suppose now that μ is a canonical Gibbs state for U at temperature $T = \sigma^2/2\lambda$; μ is a probability measure on $(\mathbf{\Omega}_0, \mathscr{R}_0)$ such that, given $\omega_V{}^c$ and the number n of particles in V, the conditional density of μ is just $f_V(\omega_V | \omega_V{}^c)$ with $h = 0$ in the definition of f_V. Let $V_n = D_0(4Rn)$ and choose the corresponding external field $h_n = h_n(x)$ in such a way that $\lim \int h_n(\omega)\, d\mu = 0$, where

$$h_n(\omega) = \sum_{i \in I} h_n(x_i(\omega))$$

Since $h_n \geqslant 0$, this implies that $Z_n = \int \exp[-T^{-1} h_n(\omega)]\, d\mu$ goes to one; thus

$$\lim \int |Z_n^{-1} \exp[-T^{-1} h_n(\omega)] - 1|\, d\mu = 0$$

follows again by the dominated convergence theorem; i.e., the probability measures μ_n defined by $d\mu_n = Z_n^{-1} \exp[-T^{-1} h_n(\omega)]\, d\mu$ converge to μ in the variation distance. On the other hand, the conditional density of μ_n with respect to the Lebesgue measure, given $\omega_{V_n}{}^c$ and the number of particles in V_n, is just $f_{V_n}(\omega_{V_n} | \omega_{V_n}{}^c)$ with $h = h_n$; therefore μ_n is invariant under the following partial dynamics: particles inside V_n are moving according to (J_{V_n}), while the position and the velocity of external particles remain the same as at time zero. Let $\mathbf{P}_t{}^n$ denote the Markov semigroup of the above partial dynamics; we have shown that $\mu_n = \mu_n \mathbf{P}_t{}^n$, i.e., $\mu_n = \mu_n \mathbf{P}_t{}^n - \mu \mathbf{P}_t{}^n + \mu \mathbf{P}_t{}^n$. We know that μ_n converges to μ in the variation distance; thus $\lim(\mu_n \mathbf{P}_t{}^n - \mu \mathbf{P}_t{}^n) = 0$, again in the variation distance. Further, as $\varphi(t, \omega, \mathbf{c}) = \lim \varphi_{V_n}(t, \omega, \mathbf{c})$ a.s.

for each $\omega \in \Omega_0$, it follows that $\lim \mu \mathbf{P}_t^n = \mu \mathbf{P}_t$ holds in the sense of weak convergence of probability measures, i.e., μ is invariant under $\mathbf{P}_t$.

To prove the "only if" part of Theorem 2, consider the time evolution of $H_f(\omega)$ along the general solution $\boldsymbol{\omega}_t = \varphi(t, \omega, \mathbf{c})$ of (I), where

$$H_f(\omega) = \sum_{i \in I} f(x_i(\omega)) H_i(\omega)$$
$$H_i(\omega) = \tfrac{1}{2}|v_i(\omega)|^2 + \tfrac{1}{2} \sum_{j \neq i} U(x_i(\omega) - x_j(\omega))$$

and $f = f(x)$, $x \in \mathbf{R}^2$, is a continuously differentiable function of compact support. Let $H_f'(\omega)$ denote the time derivative of $H_f(\omega)$ at $\omega = \bar{\omega}_t$ along the solution $\bar{\boldsymbol{\omega}}_t$ of (I) in the classical case of $\lambda = \sigma = 0$; we have

$$H_f'(\omega) = \sum_{i \in I} (\operatorname{grad} f(x_i), v_i) H_i(\omega) + \tfrac{1}{4} \sum_{i \in I} \sum_{j \neq i} [f(x_j) - f(x_i)](\operatorname{grad} U(x_i - x_j), v_i + v_i)$$

with the usual notation $x_i = x_i(\omega)$, $v_i = v_i(\omega)$; therefore

$$H_f(\omega_t) = H_f(\omega_0) + \int_0^t H_f'(\omega_s)\, ds + \sum_{i \in I} \sigma \int_0^t f(x_i) v_i\, dw_i + \sum_{i \in I} \int_0^t f(x_i)(\sigma^2 - \lambda |v_i|^2)\, ds \tag{41}$$

follows by the Ito lemma. Suppose now that μ is a canonical Gibbs state for U and temperature $T > 0$, and μ is time-invariant, i.e., $\mu_t = \mu \mathbf{P}_t = \mu$ for each $t \geqslant 0$. Then the coordinates of v_i are Gaussian random variables of zero mean and variance T; further, each v_i is independent of the positions of particles; thus $\int f(x_i(\omega))|v_i(\omega)|^2 \mu(d\omega) = 2T \int f(x_i(\omega)) \mu(d\omega)$, and $\int H_f'(\omega) \mu(d\omega) = 0$. On the other hand, the expectation of the stochastic integrals in (41) with respect to $\mathbf{P}$ is zero, since the expectation of

$$\int_0^t \sum_{i \in I} f^2(x_i(\omega_s)) |v_i(\omega_s)|^2\, ds$$

is finite in view of Proposition 1; thus, taking the expectation of both sides of (41) with respect to the product measure $\mu \times \mathbf{P}$, it follows by $\mu_t = \mu$ and the Fubini theorem that

$$(\sigma^2 - 2\lambda T) \int \sum_{i \in I} f(x_i(\omega))\, \mu(d\omega) = 0$$

for each f, which proves the last statement of Theorem 2.

7. PROOF OF THEOREM 3

Let $\boldsymbol{\omega}_t = \varphi_{\lambda,\sigma}(t, \omega, c)$, $\bar{\boldsymbol{\omega}}_t = \varphi(t, \omega)$, $\lambda > 0$, $\sigma > 0$, and $\epsilon = \max\{\lambda, \sigma\} < 1$. We have to repeat the proof of Proposition 3 in the modified situation when $\omega_0 = \bar{\omega}_0$ and $v_i = \bar{v}_i$ has a proper stochastic differential, namely

$$d(v_i - \bar{v}_i) = -[F_i(\omega_t) - F_i(\bar{\omega}_t)]\, dt - \lambda v_i\, dt + \sigma\, dw_i$$

Set $d_n(t, \rho) = d(\omega_t, \bar{\omega}_t, 0, \rho + 4Rn)$ with $\rho \geqslant 4r \geqslant 1$ fixed; then, following the lines of the proof of (26), we obtain by the Ito lemma that

$$d_n(t, \rho) \leqslant QM_1(0, \rho + 4Rn) \int_0^t d_{n+1}(s, \rho)\, ds + Z_n(t)$$

where Q is the same as in (27) and

$$Z_n(t) = \int_0^t \sum_{i \in I} f_i \bar{f}_i [2\sigma^2 - 2\lambda(v_i - \bar{v}_i, v_i)]\, ds + 2\sigma \int_0^t \sum_{i \in I} f_i \bar{f}_i (v_i - \bar{v}_i)\, dw_i \quad (42)$$

The exponential supermartingale inequality (see the proof of Lemma 2) implies that the probability of

$$\sup_{t \geqslant 0} \left[2\sigma \int_0^t \sum_{i \in I} f_i \bar{f}_i (v_i - \bar{v}_i)\, dw_i - 4\sigma \int_0^t \sum_{i \in I} f_i^2 \bar{f}_i^2 |v_i - \bar{v}_i|^2\, ds \right] \leqslant \sigma(\rho + n)^2 \quad (43)$$

exceeds $1 - \exp[-2(\rho + n)^2]$, where

$$\sum_{n=0}^{\infty} \exp[-2(\rho + n)^2] < e^{-\rho}$$

as $\rho \geqslant 1$. On the other hand, the deterministic integrals in (42) and in (43) can be estimated by a constant multiple of $\epsilon t Q^2(\rho + n)^2$; thus, combining (43) and the above inequality, we obtain that for $t \leqslant t_1$

$$\sup_{s \leqslant t} d_n(s, \rho) \leqslant QM_1(0, \rho + 4Rn) \int_0^t d_{n+1}(s, \rho)\, ds + \epsilon L_6 Q^2(\rho + n)^2 \quad (44)$$

holds simultaneously for each n with a probability larger than $1 - e^{-\rho}$; L_6 depends only on U and t_1. Iterating (44), we obtain that

$$\mathbf{P}\left[\sup_{s \leqslant t} d(\omega_s, \bar{\omega}_s, 0, \rho) \geqslant \epsilon S(\lambda, \sigma) \right] \leqslant e^{-\rho} \quad (45)$$

where the random variable $S(\lambda, \sigma)$ is given by

$$S(\lambda, \sigma) = L_6 \sum_{n=0}^{\infty} t^n M_n(0, \rho)(\rho + n)^2 Q^{n+2}$$

Further, choosing $u = 2g(n + m)$ in Proposition 1, it follows that

$$S(\lambda, \sigma) \leqslant \sum_{n=0}^{\infty} L_6 t^n M_n(0, \rho)(\rho + n)^2 \exp[q(n + 2)(d + 1/2)g(2g(n + m))]$$

holds with a probability larger than

$$1 - 2 \sum_{n=0}^{\infty} \exp[-2g(n + m)] \geqslant 1 - 2/m$$

where $q = q_w(t)$ does not depend on $\lambda \leqslant 1$ and $\sigma \leqslant 1$. Therefore the tail of the distribution of $S(\lambda, \sigma)$ is uniformly bounded, i.e., $\epsilon S(\lambda, \sigma)$ converges to zero in probability if $\epsilon = \max\{\lambda, \sigma\}$ goes to zero, so that the comparison of (45) and of Proposition 2 results in the statement of Theorem 3.

ACKNOWLEDGMENT

It is a pleasure to express my thanks to Prof. S. Goldstein for his help in revising the manuscript.

REFERENCES

1. O. E. Lanford III, The classical mechanics of one-dimensional systems of infinitely many particles, I. An existence theorem, *Commun. Math. Phys.* **9**:169 (1968); II. Kinetic theory, *Commun. Math. Phys.* **11**:257 (1969).
2. O. E. Lanford III, Time evolution of large classical systems, in *Dynamical Systems, Theory and Applications* (Lecture Notes in Phys, Vol. 38, Springer, Berlin, 1975), pp. 1–111.
3. R. L. Dobrushin and J. Fritz, Non-equilibrium dynamics of one-dimensional infinite particle systems with a hard-core interaction, *Commun. Math. Phys.* **55**:275 (1977).
4. J. Fritz and R. L. Dobrushin, Non-equilibrium dynamics of two-dimensional infinite particle systems with a singular interaction, *Commun. Math. Phys.* **57**:67 (1977).
5. R. Lang, Unendlich-dimensionale Wienerprozesse mit Wechselwirkung. I. Existenz, *Z. Wahrscheinlichkeitstheorie Verw. Gebiete* **38**:55 (1977); II. Die reversible Masse sind kanonische Gibbs–Masse, *Z. Wahrscheinlichkeitstheorie Verw. Gebiete* **39**:277 (1977).
6. H. O. Georgii, Canonical and grand canonical Gibbs states for continuum systems, *Commun. Math. Phys.* **48**:31 (1976).
7. H. P. McKean, Jr., *Stochastic Integrals* (Academic Press, New York, 1969).
8. D. Ruelle, Superstable interactions in classical statistical mechanics, *Commun. Math. Phys.* **18**:127 (1970).
9. R. L. Dobrushin, Gibbsian random fields. I, II, III, *Funkts. Anal. Pril.* **2**:31, **3**:27 (1969).